Lecture 35

10.4 - Areas and Lengths in Polar Coordinates

Lengths

Recall that if we have a polar curve $r=f(\theta)$, we can create parametric equations:

$$x=f(\theta)\cos\theta, \quad y=f(\theta)\sin\theta$$

Thus, to compute the length of the curve from $\theta=\alpha$ to $\theta=\beta$, we have

$$L=\int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2}\,d\theta$$

But, notice: $\left(r=f(\theta),\ \frac{dr}{d\theta}=f'(\theta)\right)$

$$\left(\frac{dx}{d\theta}\right)^2+\left(\frac{dy}{d\theta}\right)^2=\left[f'(\theta)\cos\theta-f(\theta)\sin\theta\right]^2+\left[f'(\theta)\sin\theta+f(\theta)\cos\theta\right]^2$$

$$=\left(\frac{dr}{d\theta}\right)^2\cos^2\theta\cancel{-2r\frac{dr}{d\theta}\cos\theta\sin\theta}+r^2\sin^2\theta+\left(\frac{dr}{d\theta}\right)^2\sin^2\theta\cancel{+2r\frac{dr}{d\theta}\sin\theta\cos\theta}+r^2\cos^2\theta$$

$$=\left(\frac{dr}{d\theta}\right)^2(\cos^2\theta+\sin^2\theta)+r^2(\sin^2\theta+\cos^2\theta)=\left(\frac{dr}{d\theta}\right)^2+r^2$$

So, we can simplify the formula a bit:

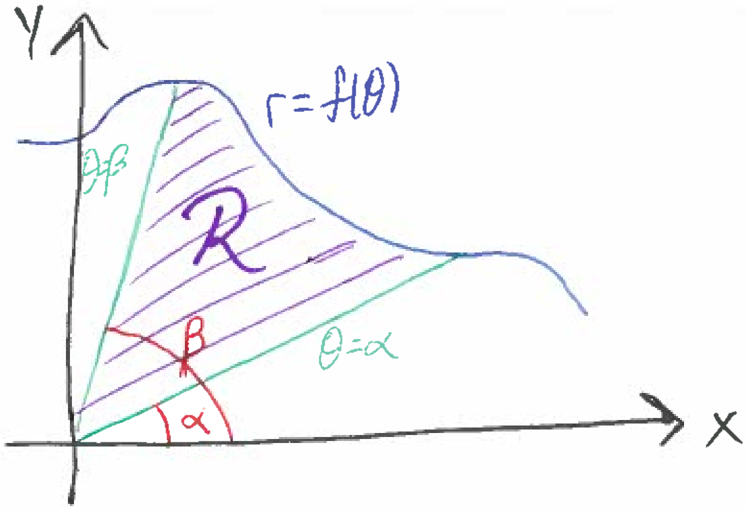
$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (r = f(\theta))$$

<u>Ex</u>: Find the length of the polar curve

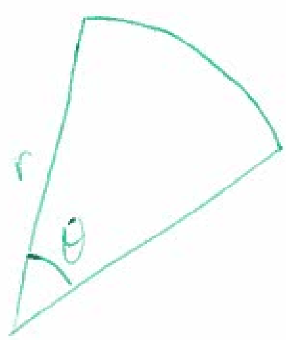
$$r = 6\sin\theta, \quad 0 \le \theta \le \pi.$$

Areas

Suppose we have a polar curve $r=f(\theta)$ and we want to know the area swept out as θ goes from α to β. One such region could look like



Just as before, we want to break this region into small pieces, except this time they won't be rectangles... but rather sectors of a circle:



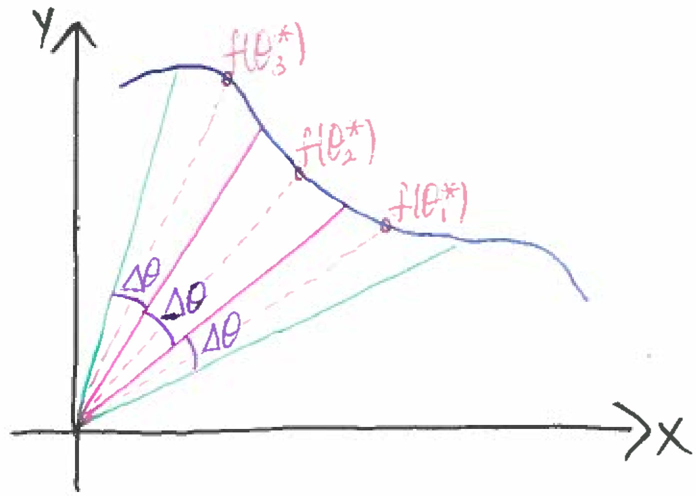
The area of this is $A=\dfrac{r^2\theta}{2}$

(This is $\frac{\theta}{2\pi}$% of a full circle, which has area πr^2.)

We divide the interval $[\alpha, \beta]$ into n subintervals of length $\Delta\theta = \frac{\beta-\alpha}{n}$, and inside each subinterval we choose a sample angle θ_i^*. Then, we can approximate the area of R as:

$$A \approx \frac{[f(\theta_1^*)]^2 \Delta\theta}{2} + \cdots + \frac{[f(\theta_n^*)]^2 \Delta\theta}{2} = \sum_{i=1}^{n} \frac{[f(\theta_i^*)]^2}{2} \Delta\theta$$

(n=3 in this picture)



Taking $n \to \infty$ in the Riemann sum above gives:

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{[f(\theta_i^*)]^2}{2} \Delta\theta_i = \int_\alpha^\beta \frac{[f(\theta)]^2}{2} d\theta = \int_\alpha^\beta \frac{r^2}{2} d\theta$$

Ex: Find the area inside one petal of the rose

$$r = \sin 2\theta$$

Sol: Step 1

Step 2

We can also find the area between to polar curves

$$r = f(\theta) \ \& \ r = g(\theta)$$

as

$$\boxed{A = \int_{\alpha}^{\beta} \frac{1}{2}\left[(f(\theta))^2 - (g(\theta))^2\right] d\theta}$$

Ex: Find the area outside the circle $r=1$ and inside the rose $r = 2\sin 2\theta$.

Because polar coordinates do not provide unique reprentations of points (unless we restrict r & θ), sometimes finding the appropriate θ-values is a little tricky.

Ex: Find the θ-values at which

$$r = 2\cos 2\theta \quad \text{and} \quad r = 1$$

intersect.